COLLOCATION COARSE APPROXIMATION (CCA) IN MULTIGRID

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Abstract. The two common approaches to defining coarse operators in multigrid numerical algorithms are discretization coarse approximation (DCA) and (Petrov-)Galerkin coarse approximation (GCA). Here, a new approach called *collocation coarse approximation* (CCA) is introduced, which—like GCA—is algebraically defined and able to cater to difficult features such as discontinuous coefficients, but, unlike GCA, allows explicit control over the coarse-grid sparsity pattern (stencil) and therefore control over the computational complexity of the solver. CCA relies on certain basis functions for which the coarse approximation to the fine-grid problem is exact. Numerical experiments for two-dimensional diffusion problems including jumping coefficients demonstrate the potential of the resulting multigrid algorithms.

Key words. iterative methods, multigrid, coarse approximation

AMS subject classifications. 65N55, 65N12, 65F10

1. Introduction. Multigrid iterative methods are well known as efficient solvers for many problems arising from the discretization of boundary-value problems (see, e.g., the introductory [7, 12, 28], the comprehensive [24], and the classical [3, 4, 15, 16].) It is common to distinguish between *geometric* multigrid, whereby the problem is associated with a regular computational grid, and *algebraic* multigrid (AMG), in which the input is typically just a matrix and a right-hand side vector of an underlying nonsingular linear system of equations

$$\mathbf{A}\mathbf{u} = \mathbf{f}\,, \tag{1.1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{f} \in \mathbb{R}^n$ are given, and $\mathbf{u} \in \mathbb{R}^n$ is the vector of unknowns. Intermediate models, which are algebraic yet require additional information, also exist, for example, AMGe [6], where access to the element stiffness matrix of an underlying finite element discretization is assumed. Multigrid algorithms generally apply two complementary processes: a simple iterative process like Jacobi or Gauss-Seidel iteration called relaxation (or smoothing), and coarse-grid correction. The coarse-grid correction should take care of those error components which cannot be efficiently reduced by the relaxation method. These slow-to-converge error components are called *algebraically* smooth and they have to be represented sufficiently well by a coarse approximation. In return, relaxation takes care of the non-smooth error components that are only resolved on the fine grid. In geometric multigrid these non-smooth components coincide with high-frequency, i.e., highly oscillating error components in accordance with the geometric notion of non-smoothness. Hence, the application of multigrid necessitates the definition of a coarse version $\mathbf{A}^c$ of $\mathbf{A}$ (actually a sequence of coarse versions, obtained recursively) which should be adjusted to the smooth error components.

The two most common approaches to defining coarse-grid operators in multigrid are *direct* or *discretization* coarse approximation and *Petrov-Galerkin* coarse-grid approximation—DCA and GCA, respectively. DCA is applied in geometric multigrid, almost exclusively for problems arising from the discretization of partial differential equations (PDEs) or systems. In this approach, $\mathbf{A}$ is some fine-grid approximation to

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a differential operator, and $\mathbf{A}^c$ is then a coarse-grid discretization of the same operator, usually employing the same discretization scheme but with a larger mesh-size. DCA enjoys the advantage of simplicity and low cost, but it is very limited in its ability to handle complex geometries, discontinuous coefficients and unstructured grids. In GCA, the coarse-grid operator is defined by the product

$$\mathbf{A}^c = \mathbf{RAP}\,, \tag{1.2}$$

where $\mathbf{R}$ and $\mathbf{P}$ denote appropriate restriction and prolongation matrices. Being just a product of matrices, this construction is algebraic, and it can therefore be applied automatically, without additional information. With a careful choice of the transfer operators—$\mathbf{R}$ and $\mathbf{P}$—and a suitable relaxation, GCA often produces a robust solver. However, it is more expensive than DCA in both computing time and memory. One drawback of GCA, which is significant in many situations, is that the control over the sparsity of $\mathbf{A}^c$ is quite limited and depends on $\mathbf{P}$ and $\mathbf{R}$. Especially for large-scale parallel computing the growth of Galerkin stencils can lead to extremely high communication overhead on coarse grids, see e.g. [8]. Thus, one must often compromise between the quality of $\mathbf{P}$ and $\mathbf{R}$, which affects the rate of convergence, and the sparsity of $\mathbf{A}^c$, which affects the computational cost.

In this paper, we propose an alternative approach for the construction of coarse approximations which we call *collocation coarse approximation* (CCA) since the basic ideas are reminiscent of collocation methods which are applied for discretizing boundary value problems. CCA is introduced and described in detail in Section 2. The main idea behind CCA is that the coarse operator is constructed in such a way that it yields an "exact approximation" of the fine grid operator w.r.t. certain (algebraically) smooth basis vectors. In principle, this approach is fully algebraic. At the same time, geometric information like the location of unknowns can easily be utilized, as demonstrated in Example 1 and worked out systematically with the help of so-called canonical basis functions. Our aim here is to eliminate the dependence of coarse-operator sparsity on the transfer operators, allowing explicit control over the sparsity pattern (also called stencil) of $\mathbf{A}^c$. And yet, we wish to do this without losing the robustness afforded by GCA, and while maintaining a reasonable computational setup cost. In fact, GCA can be considered as a special case of our approach. This interesting feature—elaborated in Section 3—motivates the use of *prolongated coarse basis functions* as a proper choice for the construction of CCA. Additional advantages of CCA, that will be explored in this and future work, are mentioned in Section 5 and demonstrated by a collection of numerical experiments presented in Section 4.

We remark that there have been previous attempts to develop methods for algebraically deriving coarse-grid operators in a non-Galerkin approach with a prescribed coarse stencil, compare with [1, 2] and, most notably with [5], but these have yet to be found practical. The use of basis vectors here for the construction of coarse approximations is similar in spirit to the frequency filtering decomposition approach [25, 26], but our goals and techniques are quite different. In [27], an approach is introduced for defining $\mathbf{A}^c$ based on Fourier analysis, that provides automatic handling and significantly greater robustness than DCA, while remaining competitive with respect to computational effort. The method is abbreviated by FCA. In this method, the coarse-operator stencil is prescribed, and its coefficients are then chosen so as to obtain compatibility between the fine and coarse operators with respect to asymptotically smooth Fourier components. While this approach is appealing and useful, it lacks a systematic handling of non-constant coefficients, and also it is not easy to

generalize to unstructured grids. Motivated by FCA, we next define our method, which aims to overcome these drawbacks, and also to yield additional flexibility. In particular, it will not be restricted by the classical geometric notion of smoothness.

2. Introducing CCA.

2.1. Motivation and main idea. To motivate and introduce our approach, we use the Full-Approximation Scheme (FAS) version of the multigrid algorithm [3, 4] to describe the coarse grid correction process. FAS does not assume linearity, but, for a linear problem such as (1.1), it is equivalent to the classical algorithm—the correction scheme. We emphasize that FAS will not be required for implementation, but it is convenient for motivating the CCA approach and it brings out its generality.

Let $\tilde{\mathbf{u}}$ denote the approximation to $\mathbf{u}$ just before the coarse-grid correction process is to be applied. Subtracting $\mathbf{A}\tilde{\mathbf{u}}$ from both sides of (1.1), we obtain the equation

$$\mathbf{A}\mathbf{u} - \mathbf{A}\tilde{\mathbf{u}} = \mathbf{f} - \mathbf{A}\tilde{\mathbf{u}} \equiv \mathbf{r}(\tilde{\mathbf{u}})\,, \tag{2.1}$$

where $\mathbf{r}(\tilde{\mathbf{u}})$ is the residual associated with the approximation $\tilde{\mathbf{u}}$. Equation (2.1) is approximated "on a coarse grid" yielding the *coarse-grid equation*,

$$\mathbf{A}^c\,\mathbf{u}^c - \mathbf{A}^c\,\hat{\mathbf{R}}\tilde{\mathbf{u}} = \mathbf{R}\mathbf{r}(\tilde{\mathbf{u}})\,, \tag{2.2}$$

where $\mathbf{R} \in \mathbb{R}^{n^c \times n}$ and $\hat{\mathbf{R}} \in \mathbb{R}^{n^c \times n}$ are given primary and secondary *restriction* matrices, respectively, $\mathbf{A}^c \in \mathbb{R}^{n^c \times n^c}$, assumed to be nonsingular, is a coarse approximation to $\mathbf{A}$, and $\mathbf{u}^c \in \mathbb{R}^{n^c}$ is the vector of coarse variables. The term "coarse" refers to the fact that n^c is chosen to be significantly smaller than n. Using (2.1), equation (2.2) can be rearranged and written as follows:

$$\mathbf{A}^c\,\mathbf{u}^c = \mathbf{f}^c + \tau^c(\tilde{\mathbf{u}})\,, \tag{2.3}$$

where $\mathbf{f}^c = \mathbf{R}\mathbf{f}$ is the right-hand side function restricted to the coarse grid, and

$$\tau^c(\tilde{\mathbf{u}}) = \left(\mathbf{A}^c\,\hat{\mathbf{R}} - \mathbf{R}\mathbf{A}\right)\tilde{\mathbf{u}}\,.$$

Upon convergence, the last term in (2.3) tends to $\tau^c(\mathbf{u})$, which is the *fine-to-coarse defect correction*, discussed below; see, e.g., [4, 17, 18]. Once $\mathbf{u}^c$ is computed, it is used to improve the fine-grid approximation as follows:

$$\tilde{\mathbf{u}}_{new} = \tilde{\mathbf{u}} + \mathbf{P}\left(\mathbf{u}^c - \hat{\mathbf{R}}\tilde{\mathbf{u}}\right)\,, \tag{2.4}$$

where $\mathbf{P} \in \mathbb{R}^{n \times n^c}$ is a given *prolongation* matrix. The coarse-grid equation (2.3) may be solved directly, yielding a two-grid method, or treated recursively, resulting in a multigrid algorithm. For simplicity, we consider the two-grid algorithm in our discussions, and later apply the usual recursive procedures to obtain multigrid algorithms in practice. We remark that for linear problems the choice of $\hat{\mathbf{R}}$ is of no consequence in existing methods because the associated terms cancel out. (In particular, setting $\hat{\mathbf{R}} = 0$ produces the classical linear—Correction Scheme—multigrid algorithm.) In our approach, however, $\hat{\mathbf{R}}$ will influence the construction of $\mathbf{A}^c$.

Consider the "role" of τ^c in (2.3). Without this term, the coarse equation is simply a coarse approximation to the original system. The addition of this term in the right-hand side causes (at convergence, whereupon $\mathbf{r} = \mathbf{0}$ and $\tilde{\mathbf{u}} = \mathbf{u}$) the coarse solution to be equal to the sought fine solution, restricted to the coarse grid: $\mathbf{u}^c = \hat{\mathbf{R}}\mathbf{u}$. This is immediately seen in (2.2), by setting $\mathbf{r} = \mathbf{0}$ and $\tilde{\mathbf{u}} = \mathbf{u}$. Loosely speaking, $\tau^c(\mathbf{u})$ is the term that we need to add to the coarse right-hand side so as to obtain fine-grid accuracy on the coarse grid. Suppose now that we could construct a coarse matrix, $\mathbf{A}^c$, such that $\tau^c \equiv \mathbf{0}$ independently of $\tilde{\mathbf{u}}$. Then, since $\tau^c(\tilde{\mathbf{u}}) = \tau^c(\mathbf{u})$ for any $\tilde{\mathbf{u}}$, the solution to the coarse-grid equation satisfies $\mathbf{u}^c = \hat{\mathbf{R}}\mathbf{u}$ regardless of $\tilde{\mathbf{u}}$. This yields the best possible coarse grid correction. In particular, given such a $\mathbf{u}^c$, we can often reconstruct $\mathbf{u}$ by solving a problem that is smaller and much better conditioned than the original fine-grid problem (associated with so-called compatible relaxation, introduced in [5] and developed and applied in, e.g., [13, 14, 19, 20].) A particularly well-known case of this occurs when the coarse-grid variables are defined to be a subset of the fine-grid variables, and $\hat{\mathbf{R}}$ is the simple injection operator. In this case, if $\tau^c \equiv 0$, then we obtain an exact solution at coarse-grid points. The associated $\mathbf{A}^c$ is the so-called Schur complement. Unfortunately, except in special cases (like tri-diagonal matrices or certain variants of total reduction, cf. [23]), such a matrix $\mathbf{A}^c$ is much denser than $\mathbf{A}$. Since maintaining sparsity is crucial for an efficient multigrid solver, we are motivated to prescribe the sparsity pattern of $\mathbf{A}^c$, and then choose for each coarse equation a particular set of *basis functions* for which the corresponding element of τ^c should vanish. The number of basis functions in the set depends on the number of nonzeros allowed by the prescribed sparsity pattern in the corresponding row of the matrix $\mathbf{A}^c$. This approach is elaborated next.

2.2. Explicit construction of the CCA operator. To demonstrate the broad range of applicability of CCA we choose a rather general description. However, in an effort to clarify the presentation, this description is accompanied by a simple example (Example 1) which will be used to explain the CCA process in detail. Throughout this paper we restrict our examples to scalar PDEs in two dimensions, discretized on structured grids, employing the standard stencil notation for clarity. More general applications will be explored in future work.

To construct the ith row of the coarse matrix $\mathbf{A}^c$—denoted $\mathbf{A}^c_i$ $(i = 1, \ldots, n^c)$—we first need to prescribe its sparsity pattern (stencil), i.e., the set of j's for which the matrix elements $\mathbf{A}^c_{ij}$ may be nonzero. We denote this "neighborhood" of point i by $\Omega^{(i)} = \{\omega^{(i)}_1, \ldots, \omega^{(i)}_{n^{(i)}}\}$, and the number of elements in the set by $n^{(i)} = |\Omega^{(i)}|$. The sparsity requirement implies that $n^{(i)}$ is very much smaller than n^c, typically bounded by a small constant.

The stencil of $\mathbf{A}^c_i$ can be prescribed by a given matrix $\mathbf{T}^{(i)}$ of size $n^{(i)} \times n^c$ that acts as a simple restriction from the coarse grid to the neighborhood $\Omega^{(i)}$. Each row of $\mathbf{T}^{(i)}$ corresponds to a different element of $\Omega^{(i)}$ as follows:

$$\mathbf{T}^{(i)}_{k,j} = \begin{cases} 1 & \text{for } j = \omega^{(i)}_k\,, \\ 0 & \text{otherwise}\,, \end{cases} \qquad k = 1, \ldots, n^{(i)}, \quad j = 1, \ldots, n^c\,. \tag{2.5}$$

Obviously, in each row of $\mathbf{T}^{(i)}$ only one element is equal to 1 and the remaining elements are zero.

EXAMPLE 1. *Consider the common five-point star stencil (possibly modified at*

boundaries) often used in the discretization of two-dimensional diffusion operators:

$$\begin{bmatrix} & N_{\mathbf{i}} & \\ W_{\mathbf{i}} & C_{\mathbf{i}} & E_{\mathbf{i}} \\ & S_{\mathbf{i}} & \end{bmatrix}_h \qquad (\mathbf{i} = 1, \ldots, n) \ . \tag{2.6}$$

We assume for simplicity a square fine grid of mesh-size $h = 1/n_x = 1/n_y$ *with* $n_x = n_y$ *even. After elimination of boundary conditions we end up with a linear system of* $(n_x - 1) \times (n_y - 1) = n$ *equations, see (1.1). Employing a lexicographic ordering of grid points, the eight direct fine grid neighbors of an inner grid point* $\mathbf{i}$ *have indices* $\mathbf{i} - n_x$ *(southwestern),* $\mathbf{i} - n_x + 1$ *(southern),* $\mathbf{i} - n_x + 2$ *(southeastern),* $\mathbf{i} - 1$ *(western),* $\mathbf{i} + 1$ *(eastern),* $\mathbf{i} + n_x - 2$ *(northwestern),* $\mathbf{i} + n_x - 1$ *(northern), and* $\mathbf{i} + n_x$ *(northeastern), see Figure 2.1.*

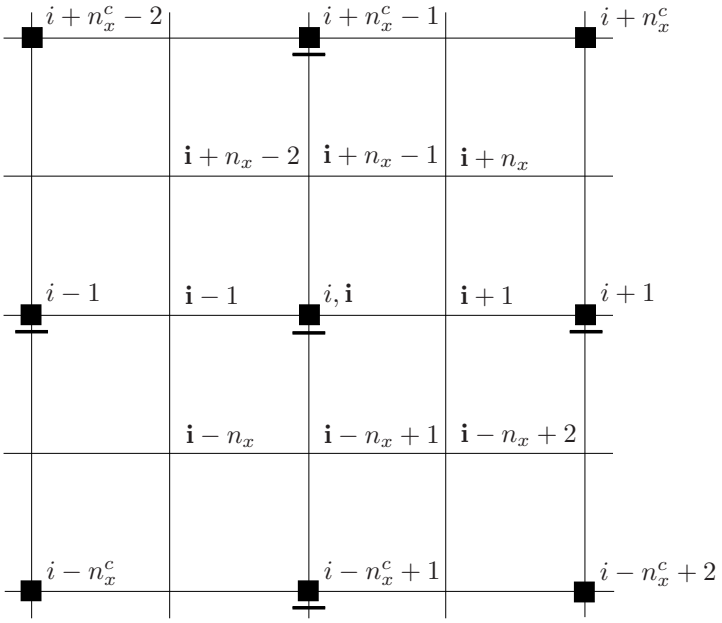

FIG. 2.1. *Direct neighbors of coarse grid point* i *and co-located fine grid point* $\mathbf{i}$*. Coarse variables (C points) are marked by* ■*.*

We assume standard coarsening obtained by doubling the meshsize yielding $n^c = (n_x/2 - 1) \times (n_y/2 - 1)$*. Note that the coarse variables are "co-located" with a subsect of fine variables, see Figure 2.1. Boldface indices as used in (2.6) designate fine-grid elements, with the convention that if* i *is the index of a coarse variable, then* $\mathbf{i}$ *denotes the index of the co-located fine-grid variable. The eight direct neighbors of a coarse grid point* i *are characterized analogously as above for a fine grid point* $\mathbf{i}$ *by simply replacing* n_x, n_y *with* $n_x^c = n_x/2, n_y^c = n_y/2$ *and* $\mathbf{i}$ *with* i*, respectively (cf. Figure 2.1).*

On the coarse grid we prescribe the same five-point star stencil as the one used on the fine grid, cf. (2.6), with $n^{(i)} = 5$*. This means that the neighborhood of coarse grid point* i *is given by* $\Omega^{(i)} = \{i - n_x^c + 1, i - 1, i, i + 1, i + n_x^c - 1\}$*. The corresponding points are underlined in Figure 2.1.* $\mathbf{T}^{(i)} \in \mathbb{R}^{5 \times n^c}$ *from (2.5) has only five nonzero entries given by*

$$\mathbf{T}^{(i)}_{1,i-n_x^c+1} = \mathbf{T}^{(i)}_{2,i-1} = \mathbf{T}^{(i)}_{3,i} = \mathbf{T}^{(i)}_{4,i+1} = \mathbf{T}^{(i)}_{5,i+n_x^c-1} = 1 \ . \tag{2.7}$$

We assume that the restriction matrices, $\mathbf{R}$ (whose ith row is denoted $\mathbf{R}_i$) and $\hat{\mathbf{R}}$, are prescribed. Next, we choose a set of $n^{(i)}$ suitable basis functions $\mathbf{b}_k^{(i)} \in \mathbb{R}^n$ $(k = 1, \ldots, n^{(i)})$. $\mathbf{A}_i^c$ is now defined by imposing that the ith element of $\tau^c\left(\mathbf{b}_k^{(i)}\right)$ vanish for each of the basis functions:

$$\tau_i^c\left(\mathbf{b}_k^{(i)}\right) = \left(\mathbf{A}_i^c\,\hat{\mathbf{R}} - \mathbf{R}_i\mathbf{A}\right)\mathbf{b}_k^{(i)} = 0\,, \quad k = 1, \ldots, n^{(i)}\,, \tag{2.8}$$

subject to the prescribed stencil. This yields a linear system of $n^{(i)}$ equations in the $n^{(i)}$ unknown coefficients of $\mathbf{A}_i^c$. To write the system explicitly, we denote by

$$\mathbf{B}^{(i)} \equiv \left[\mathbf{b}_1^{(i)}\,\mathbf{b}_2^{(i)}\,\ldots\,\mathbf{b}_{n^{(i)}}^{(i)}\right] \tag{2.9}$$

the matrix of size n by $n^{(i)}$, whose columns are the basis functions, and by

$$\mathbf{M}^{(i)} \equiv \mathbf{T}^{(i)}\hat{\mathbf{R}}\mathbf{B}^{(i)}\,, \tag{2.10}$$

the $n^{(i)} \times n^{(i)}$ matrix whose columns are obtained by restricting the basis functions to the coarse grid and then sampling them on $\Omega^{(i)}$ by $\mathbf{T}^{(i)}$. Let $\mathbf{v}^{(i)} = \mathbf{T}^{(i)}\left(\mathbf{A}_i^c\right)^T$ denote a column vector of size $n^{(i)}$ comprised of the nonzero coefficients of $\mathbf{A}_i^c$ that we need to determine. Then, by (2.8), (2.9), and (2.10), $\mathbf{v}^{(i)}$ is the solution to the linear system

$$\left(\mathbf{M}^{(i)}\right)^T\mathbf{v}^{(i)} = \left(\mathbf{R}_i\mathbf{A}\mathbf{B}^{(i)}\right)^T\,. \tag{2.11}$$

Once $\mathbf{v}^{(i)}$ is computed, the ith row of $\mathbf{A}^c$ is obtained from

$$\mathbf{A}_i^c = \left(\mathbf{v}^{(i)}\right)^T\mathbf{T}^{(i)}\,. \tag{2.12}$$

Evidently, $\mathbf{A}_i^c$ is uniquely defined if and only if the restricted basis functions are linearly independent with respect to $\Omega^{(i)}$, i.e., $\mathbf{M}^{(i)}$ is nonsingular. We henceforth assume that the basis functions are chosen accordingly. Given this, we obtain from (2.11) and (2.12) the following explicit relation:

$$\mathbf{A}_i^c = \mathbf{R}_i\mathbf{A}\mathbf{B}^{(i)}\left(\mathbf{M}^{(i)}\right)^{-1}\mathbf{T}^{(i)}\,. \tag{2.13}$$

Summarizing, the construction of the CCA operator proceeds as follows.

Fix restriction operators $\mathbf{R}$ and $\hat{\mathbf{R}}$.
Then, for each coarse point $i \in \{1, \ldots, n^c\}$ do:

1. Fix a coarse grid stencil of your choice with $n^{(i)}$ elements (i.e., the indices of the nonzero entries in the ith row of $\mathbf{A}^c$).
2. Select $n^{(i)}$ local basis functions $\mathbf{b}_k^{(i)}$.
3. Construct the coarse grid operator such that the basis functions are approximated exactly on the coarse grid, i.e.

$$\mathbf{A}_i^c\hat{\mathbf{R}}\mathbf{b}_k^{(i)} = \mathbf{R}_i\mathbf{A}\mathbf{b}_k^{(i)} \qquad (k = 1, \ldots, n^{(i)})\,, \tag{2.14}$$

cf. (2.8). This gives rise to a small $(n^{(i)} \times n^{(i)})$ linear system for the nonzero entries in $\mathbf{A}_i^c$, see (2.11).

EXAMPLE 1 CONTINUED. *Motivated by the asymptotically smooth Fourier modes of [27], we choose as our five basis functions,* $\{\mathbf{b}_1^{(i)},\dots,\mathbf{b}_5^{(i)}\}$*, the low-order monomials,* $\{1,x,x^2,y,y^2\}$*, sampled on the grid. Without loss of generality, we set* $h=1$ *and shift the origin to the mesh point* i*. The five basis functions (restricted to the subdomain relevant to* $\mathbf{A}_i^c$ *and reshaped to 2D form) are given by :*

$$\begin{bmatrix} 1 & 1 & \underline{\mathbf{1}} & 1 & 1\\ 1 & 1 & 1 & 1 & 1\\ \underline{\mathbf{1}} & 1 & \underline{\mathbf{1}} & 1 & \underline{\mathbf{1}}\\ 1 & 1 & 1 & 1 & 1\\ 1 & 1 & \underline{\mathbf{1}} & 1 & 1 \end{bmatrix}_h, \quad \begin{bmatrix} -2 & -1 & \underline{\mathbf{0}} & 1 & 2\\ -2 & -1 & 0 & 1 & 2\\ -\underline{\mathbf{2}} & -1 & \underline{\mathbf{0}} & 1 & \underline{\mathbf{2}}\\ -2 & -1 & 0 & 1 & 2\\ -2 & -1 & \underline{\mathbf{0}} & 1 & 2 \end{bmatrix}_h, \quad \begin{bmatrix} 4 & 1 & \underline{\mathbf{0}} & 1 & 4\\ 4 & 1 & 0 & 1 & 4\\ \underline{\mathbf{4}} & 1 & \underline{\mathbf{0}} & 1 & \underline{\mathbf{4}}\\ 4 & 1 & 0 & 1 & 4\\ 4 & 1 & \underline{\mathbf{0}} & 1 & 4 \end{bmatrix}_h,$$

$$\begin{bmatrix} 2 & 2 & \underline{\mathbf{2}} & 2 & 2\\ 1 & 1 & 1 & 1 & 1\\ \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}}\\ -1 & -1 & -1 & -1 & -1\\ -2 & -2 & -\underline{\mathbf{2}} & -2 & -2 \end{bmatrix}_h, \quad \begin{bmatrix} 4 & 4 & \underline{\mathbf{4}} & 4 & 4\\ 1 & 1 & 1 & 1 & 1\\ \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}}\\ 1 & 1 & 1 & 1 & 1\\ 4 & 4 & \underline{\mathbf{4}} & 4 & 4 \end{bmatrix}_h, \tag{2.15}$$

where elements corresponding to coarse variables co-located with members of the coarse grid stencil $\Omega^{(i)}$ *are written in boldface and underlined. Let* $\hat{\mathbf{R}}$ *refer to simple injection. Then, (2.11) reads:*

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1\\ 0 & -2 & 0 & 2 & 0\\ 0 & 4 & 0 & 4 & 0\\ -2 & 0 & 0 & 0 & 2\\ 4 & 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1^{(i)}\\ \mathbf{v}_2^{(i)}\\ \mathbf{v}_3^{(i)}\\ \mathbf{v}_4^{(i)}\\ \mathbf{v}_5^{(i)} \end{pmatrix} = \begin{pmatrix} \mathbf{g}_1^{(i)}\\ \mathbf{g}_2^{(i)}\\ \mathbf{g}_3^{(i)}\\ \mathbf{g}_4^{(i)}\\ \mathbf{g}_5^{(i)} \end{pmatrix}, \tag{2.16}$$

where $\mathbf{g}_k^{(i)} = \mathbf{R}_i\mathbf{A}\mathbf{b}_k^{(i)}$ *is defined by applying the fine-grid operator to the* k*th basis function, and restricting to the coarse-grid point* i *using the prescribed restriction,* $\mathbf{R}$*. Here, in going from (2.15) to (2.16),* $\mathbf{T}^{(i)}$ *has been defined such that the ordering of the variables is* S_i*,* W_i*,* C_i*,* E_i*,* N_i*, i.e.,* $\mathbf{v}^{(i)} = \begin{pmatrix} S_i & W_i & C_i & E_i & N_i \end{pmatrix}^T$*, corresponding to the lexicographic ordering of grid points, see Figure 2.1 and* $\mathbf{T}^{(i)}$ *from (2.7).*

The stencil elements referring to the coarse grid point i *can be easily obtained from (2.16). They read*

$$\mathbf{v}^{(i)} = \begin{pmatrix} S_i\\ W_i\\ C_i\\ E_i\\ N_i \end{pmatrix} = \begin{pmatrix} -\mathbf{g}_4^{(i)}/4 + \mathbf{g}_5^{(i)}/8\\ -\mathbf{g}_2^{(i)}/4 + \mathbf{g}_3^{(i)}/8\\ \mathbf{g}_1^{(i)} - \mathbf{g}_3^{(i)}/4 - \mathbf{g}_5^{(i)}/4\\ \mathbf{g}_2^{(i)}/4 + \mathbf{g}_3^{(i)}/8\\ \mathbf{g}_4^{(i)}/4 + \mathbf{g}_5^{(i)}/8 \end{pmatrix}.$$

Equation (2.14) establishs a bridge to collocation methods and motivates the name of our approach. Collocation is a well-known technique for the discretization of boundary value problems (BVP), see, e.g., [21, 22]. Here, a set of n^c linearly independent basis functions and n^c different collocation points are prescribed. The collocation points may be located at the boundary or in the interior of the underlying domain. Then, the continuous solution of the BVP is approximated by a linear combination of the basis functions in such a way that the approximation is an exact

solution to the BVP restricted to the prescribed n^c collocation points. The connection to CCA becomes apparent by interpreting the BVP as the "fine problem" and the approximation obtained by collocation as the "coarse approximation"—which has fine-grid accuracy at the n^c "collocation" points, i.e., the n^c coarse points. In fact, CCA involves a certain generalization of the collocation idea since we may use different basis functions at different "collocation" points.

The explicit formula (2.13) for the coarse grid operator—which can lead to handy expressions for the coarse grid stencil entries as demonstrated in Example 1—motivates the definition of canonical basis functions in the next subsection. Here, explicit formulas for the stencil elements are precomputed in a systematic manner, avoiding the computational expense of solving the small linear systems (2.11) for each coarse grid unknown.

2.3. Canonical Basis Functions. Evidently, determining $\mathbf{A}_i^c$ requires solving a linear system of $n^{(i)}$ equations and unknowns. If the grid is structured and the same set of basis functions (after shifting the origin to the point i) is used for all (or at least many) of the coarse variables, we can eliminate this computational overhead by replacing the basis functions by an equivalent set of *canonical basis functions* (CBFs) as we next describe. The idea, motivated by (2.13), is to choose a set of basis functions that spans the same subspace as the columns of $\mathbf{B}^{(i)}$, such that $\mathbf{M}^{(i)}$ in (2.10) will be the identity matrix.

Define $\mathbf{B}^{(i)}$ and $\mathbf{M}^{(i)}$ as in (2.9) and (2.10), and assume that $\mathbf{M}^{(i)}$ is nonsingular. Then, the CBFs are defined as follows.

DEFINITION 2.1. *The canonical basis functions corresponding to the ith coarse variable are the columns of the matrix* $\bar{\mathbf{B}}^{(i)}$*, defined by*

$$\bar{\mathbf{B}}^{(i)} = \mathbf{B}^{(i)} \left(\mathbf{M}^{(i)} \right)^{-1} .$$

The CBFs satisfy the following uniqueness and sparseness properties.

PROPOSITION 2.2. *All sets of basis functions that span the same linear subspace, and are linearly independent on* $\Omega^{(i)}$*, produce the same set of canonical basis functions.*

Proof. Let $\mathbf{B}^{(i)}$ be a matrix of size n by $n^{(i)}$, whose columns form a set of basis functions, and let $\mathbf{B}'^{(i)}$ be some other matrix of the same size and of the same range as $\mathbf{B}^{(i)}$. Then, $\mathbf{B}'^{(i)}$ can be written as

$$\mathbf{B}'^{(i)} = \mathbf{B}^{(i)} \mathbf{Q} ,$$

where $\mathbf{Q}$ is some nonsingular matrix of size $n^{(i)}$ by $n^{(i)}$. Following (2.10), set

$$\mathbf{M}'^{(i)} = \mathbf{T}^{(i)} \hat{\mathbf{R}} \mathbf{B}'^{(i)} = \mathbf{M}^{(i)} \mathbf{Q} .$$

Now, the canonical form of $\mathbf{B}'^{(i)}$ is given by

$$\begin{aligned} \bar{\mathbf{B}}'^{(i)} &= \mathbf{B}'^{(i)} \left(\mathbf{M}'^{(i)} \right)^{-1} = \mathbf{B}^{(i)} \mathbf{Q} \left(\mathbf{M}^{(i)} \mathbf{Q} \right)^{-1} = \mathbf{B}^{(i)} \mathbf{Q} \mathbf{Q}^{-1} \left(\mathbf{M}^{(i)} \right)^{-1} \\ &= \mathbf{B}^{(i)} \left(\mathbf{M}^{(i)} \right)^{-1} = \bar{\mathbf{B}}^{(i)} . \end{aligned}$$

□

PROPOSITION 2.3. *The matrix $\bar{\mathbf{M}}^{(i)}$, obtained from (2.10) using $\bar{\mathbf{B}}^{(i)}$ instead of $\mathbf{B}^{(i)}$ is the $n^{(i)}$ by $n^{(i)}$ identity matrix.*

Proof. By (2.10) and Definition 2.1

$$\bar{\mathbf{M}}^{(i)} = \mathbf{T}^{(i)}\hat{\mathbf{R}}\mathbf{B}^{(i)}\left(\mathbf{M}^{(i)}\right)^{-1} = \mathbf{M}^{(i)}\left(\mathbf{M}^{(i)}\right)^{-1}.$$

□

As a result of this, to compute the kth nonzero coefficient of $\mathbf{A}_i^c$ we simply need to apply the fine-grid operator $\mathbf{A}$ to the kth canonical basis function, $\bar{\mathbf{b}}_k^{(i)}$, and apply the restriction $\mathbf{R}_i$. This yields $\mathbf{A}_{ij}^c$ explicitly, where j is the coarse index associated with the kth CBF, i.e., $\bar{\mathbf{b}}_k^{(i)} = \bar{\mathbf{b}}_{k(j)}^{(i)}$. Equivalently, by (2.13) and Definition 2.1 we have:

$$\mathbf{A}_i^c = \mathbf{R}_i\mathbf{A}\bar{\mathbf{B}}^{(i)}\mathbf{T}^{(i)}, \tag{2.17}$$

and therefore,

$$\mathbf{A}_{ij}^c = \mathbf{R}_i\mathbf{A}\bar{\mathbf{b}}_{k(j)}^{(i)}. \tag{2.18}$$

EXAMPLE 1 CONTINUED. *Using the basis functions corresponding to $\{1, x, x^2, y, y^2\}$, the canonical basis functions (reshaped to the 2D style) turn out to be:*

$$\bar{\mathbf{b}}_1^{(i)} = \frac{1}{4}\begin{bmatrix} 0 & -1 & \underline{\mathbf{0}} & -1 & 0 \\ -1 & 2 & 3 & 2 & -1 \\ \underline{\mathbf{0}} & 3 & \underline{\mathbf{4}} & 3 & \underline{\mathbf{0}} \\ -1 & 2 & 3 & 2 & -1 \\ 0 & -1 & \underline{\mathbf{0}} & -1 & 0 \end{bmatrix}_h, \quad \bar{\mathbf{b}}_2^{(i)} = \frac{1}{8}\begin{bmatrix} 8 & 3 & \underline{\mathbf{0}} & -1 & 0 \\ 8 & 3 & 0 & -1 & 0 \\ \underline{\mathbf{8}} & 3 & \underline{\mathbf{0}} & -1 & \underline{\mathbf{0}} \\ 8 & 3 & 0 & -1 & 0 \\ 8 & 3 & \underline{\mathbf{0}} & -1 & 0 \end{bmatrix}_h, \tag{2.19}$$

with $\bar{\mathbf{b}}_3^{(i)}$, $\bar{\mathbf{b}}_4^{(i)}$, and $\bar{\mathbf{b}}_5^{(i)}$, given by successive 90° rotations of $\bar{\mathbf{b}}_2^{(i)}$. Here, the elements at locations coinciding with the nonzero elements of $\mathbf{A}_i^c$ are again underlined and written in boldface digits.

Note that the sparsity pattern of the CBFs (2.19) is consistent with Proposition 2.3, that is, only one underlined element of $\bar{\mathbf{b}}_k^{(i)}$ $(k = 1, \ldots, 5)$ is equal to one whereas the remaining underlined elements are zero.

EXAMPLE 2. *Consider a nine-point stencil at coarse grid point i,*

$$\begin{bmatrix} Nw_i & N_i & Ne_i \\ W_i & C_i & E_i \\ Sw_i & S_i & Se_i \end{bmatrix}_{2h}. \tag{2.20}$$

A natural choice for the nine basis functions is $\{1, x, x^2\} \times \{1, y, y^2\}$, with the origin again shifted to the location of the ith coarse variable. This choice yields the following equivalent CBFs:

$$\bar{\mathbf{b}}_1^{(i)} = \frac{1}{16}\begin{bmatrix} \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} \\ 0 & 9 & 12 & 9 & 0 \\ \underline{\mathbf{0}} & 12 & \underline{\mathbf{16}} & 12 & \underline{\mathbf{0}} \\ 0 & 9 & 12 & 9 & 0 \\ \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} \end{bmatrix}_h, \quad \bar{\mathbf{b}}_2^{(i)} = \frac{1}{32}\begin{bmatrix} \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} \\ 24 & 9 & 0 & -3 & 0 \\ \underline{\mathbf{32}} & 12 & \underline{\mathbf{0}} & -4 & \underline{\mathbf{0}} \\ 24 & 9 & 0 & -3 & 0 \\ \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} \end{bmatrix}_h,$$

$$\bar{\mathbf{b}}_6^{(i)} = \frac{1}{64}\begin{bmatrix} \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} \\ -8 & -3 & 0 & 1 & 0 \\ \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} \\ 24 & 9 & 0 & -3 & 0 \\ \underline{\mathbf{64}} & 24 & \underline{\mathbf{0}} & -8 & \underline{\mathbf{0}} \end{bmatrix}_h, \tag{2.21}$$

with $\bar{\mathbf{b}}_3^{(i)}$, $\bar{\mathbf{b}}_4^{(i)}$, *and* $\bar{\mathbf{b}}_5^{(i)}$, *given by successive* $90°$ *rotations of* $\bar{\mathbf{b}}_2^{(i)}$, *and with* $\bar{\mathbf{b}}_7^{(i)}$, $\bar{\mathbf{b}}_8^{(i)}$, *and* $\bar{\mathbf{b}}_9^{(i)}$, *given by successive* $90°$ *rotations of* $\bar{\mathbf{b}}_6^{(i)}$. *Note again that the sparsity pattern is consistent with Proposition 2.3 if* $\hat{\mathbf{R}}$ *is chosen to be the injection operator.*

2.4. Error propagation by CCA coarse grid correction. For the remainder, we specialize in the common case where $\hat{\mathbf{R}}$ is the simple injection operator, implying that the coarse variables are "co-located" with a subset of the fine variables. Although this choice of coarsening is very common (the most prominent case, i.e., standard coarsening in connection with geometric multigrid, appears in Example 1), other choices for $\hat{\mathbf{R}}$ are preferred in some cases, in particular for cell-centered or staggered discretizations. Furthermore, we adopt the following conventions.

1. The fine-grid variables that are co-located with the coarse variables are called C-variables, while the complement variables are called F-variables. C and F denote the sets of indices of C- and F-variables, respectively.
2. As already introduced in Example 1, we denote the index of a fine-grid C-variable that is co-located with the coarse variable i in boldface, i.e., by $\mathbf{i}$.
3. $\hat{\mathbf{R}}$ is simple injection, that is, for $i = 1, \ldots, n^c$, $\hat{\mathbf{R}}_{i,\mathbf{i}} = 1$, and $\hat{\mathbf{R}}_{i,\mathbf{j}} = 0$ for all $\mathbf{j} \neq \mathbf{i}$.
4. The prolongation matrix is an interpolation, satisfying $\mathbf{P}_{\mathbf{i},i} = 1$ for all $i = 1, \ldots, n^c$, and $\mathbf{P}_{\mathbf{i},j} = 0$ for all $j \neq i$. That is, at C-points the coarse-grid correction is injected from the co-located coarse-grid point.

DEFINITION 2.4. *Given a matrix,* $\mathbf{B}$, *let* $\mathcal{R}(\mathbf{B})$ *denote the range of* $\mathbf{B}$, *defined as the subspace spanned by its columns. We say that a vector* $\mathbf{x}$ *is in the range of* $\mathbf{B}$, *denoted by* $\mathbf{x} \in \mathcal{R}(\mathbf{B})$, *if* $\mathbf{x}$ *can be expressed as a linear combination of the columns of* $\mathbf{B}$.

Let $\mathbf{e} = \mathbf{u} - \tilde{\mathbf{u}}$ and $\mathbf{e}_{new} = \mathbf{u} - \tilde{\mathbf{u}}_{new}$, respectively, denote the error in the approximation before and after the two-grid cycle described in Section 2.1. We have noted above that, in the special case where $\tau^c \equiv \mathbf{0}$, $\mathbf{e}_{new}$ is zero at C points regardless of $\mathbf{e}$. We now observe that, for the CCA two-grid algorithm, $\mathbf{e}_{new}$ vanishes at C-points provided that $\mathbf{e}$ is in the range of every one of the basis-function matrices, $\mathbf{B}^{(i)}$, $i = 1, \ldots, n^c$.

PROPOSITION 2.5. *Let* $\mathbf{B}^{(i)}$, $i = 1, \ldots, n^c$, *denote the matrices whose columns are the basis functions employed in the* CCA *algorithm. Let* $\mathbf{e}$, *the fine-grid error before the* CCA *coarse-grid correction, satisfy* $\mathbf{e} \in \cap_{i=1}^{n^c} \mathcal{R}\left(\mathbf{B}^{(i)}\right)$. *Then* $\mathbf{e}_{new}$, *the error after the coarse-grid correction, satisfies* $(\mathbf{e}_{new})_{\mathbf{i}} = 0$ *for all* $\mathbf{i} \in C$.

Proof. Since $\mathbf{e}$ can be expressed as a linear combination of the columns of $\mathbf{B}^{(i)}$ for each coarse grid point i, (2.8) implies that $\tau^c(\mathbf{e}) \equiv \mathbf{0}$. This gives

$$\tau^c(\tilde{\mathbf{u}}) = \tau^c(\mathbf{u}) - \tau^c(\mathbf{e}) = \tau^c(\mathbf{u}) \ .$$

Applying (2.3) we see that $\mathbf{u}^c = \hat{\mathbf{R}}\mathbf{u}$. Thus, $\mathbf{u}^c_i = \mathbf{u}_\mathbf{i}$ for all i, and (2.4) implies that $(\tilde{\mathbf{u}}_{new})_\mathbf{i} = \mathbf{u}^c_i = \mathbf{u}_\mathbf{i}$ for all $\mathbf{i} \in C$. ∎

EXAMPLE 1 CONTINUED. *Observe that, by Proposition 2.5, if the error just before the coarse-grid correction in Example 1 is given by a second-order polynomial, which excludes the mixed term xy, then this error is eliminated at all C-points by the two-level coarse-grid correction.*

Proposition 2.5 implies exact coarse-grid correction at C points only for a low-dimensional subspace (spanned by five vectors in our example.) Note, however, that when $\mathbf{A}$, $\mathbf{A}^c$, $\hat{\mathbf{R}}$ and $\mathbf{R}$ are sparse, then only a small number of elements of each basis function is significant in determining the coarse stencil. In Example 1 we have 21 significant elements: the 25 elements shown in each basis function in (2.15) minus the four corner elements, which play no part when a five-point star stencil is used. In other words, the five basis vectors can only be selected from a subspace spanned by 21 vectors that are linearly independent on the 21 grid points that are relevant for the determination of the coarse stencil. The rest of the elements in each basis function may be classified as "don't care", and therefore each basis function actually represents a large set of vectors. In most relevant problems, particularly PDEs and PDE systems, the significant elements of basis functions $\mathbf{b}^{(i)}_k$ are limited to a small local neighborhood around C-variable i. Thus, generally speaking, all errors that are locally well-approximated by the sets of basis functions will have small τ^c values, and hence will be corrected very well (though not exactly) by the coarse-grid correction process. Summarizing, the quality of the CCA coarse grid approximation for each C-variable i depends on how well the set of basis vectors locally represents the entire space of vectors that are linearly independent on those grid points that are significant for the construction of the coarse stencil.

3. Prolongated basis functions. In Example 1 *geometrically* smooth basis functions have been applied for simplicity. However, these functions may not turn out to be *algebraically* smooth, which is an essential feature in order to obtain good coarse approximations in general situations. Hence, a crucial question that remains to be answered is how to choose the basis functions appropriately. A promising choice is to use *prolongated coarse basis functions* which can be motivated by comparing CCA and GCA.

In GCA, by (1.2), the (i,j)th element of the coarse-grid operator is given by:

$$\mathbf{A}^c_{ij} = \mathbf{R}_i \mathbf{A} \mathbf{P}^j \ , \tag{3.1}$$

where $\mathbf{P}^j$ is the jth column of $\mathbf{P}$. If we now compare (3.1) to (2.18) we find that GCA is a particular case of CCA in which the canonical basis function used for the construction of the element $\mathbf{A}^c_{ij}$ is the jth column of $\mathbf{P}$ (and the stencil is prescribed to be that of the GCA operator.) Since $\mathbf{P}^j$ can be considered as the prolongation of a coarse grid function that vanishes everywhere except at the coarse grid point j, we are motivated to consider basis functions that are in the range of the prolongation, i.e., prolongations of some coarse-grid basis functions $\mathbf{b}^{c,(i)}_k$:

$$\mathbf{b}^{(i)}_k = \mathbf{P}\mathbf{b}^{c,(i)}_k \qquad (k = 1, \ldots, n^{(i)}) \ .$$

Substitution into the CCA coarse operator constraint (2.14) yields at each coarse point $i = 1, \ldots, n^c$,

$$\mathbf{A}_i^c \hat{\mathbf{R}} \mathbf{P} \mathbf{b}_k^{c,(i)} = \mathbf{R}_i \mathbf{A} \mathbf{P} \mathbf{b}_k^{c,(i)} \qquad (k = 1, \ldots, n^{(i)}) \ .$$

Now, if $\hat{\mathbf{R}}\mathbf{P} = \mathbf{I}^c$, where $\mathbf{I}^c$ is the coarse-grid identity matrix, then we obtain

$$\mathbf{A}_i^c \mathbf{b}_k^{c,(i)} = \mathbf{R}_i \mathbf{A} \mathbf{P} \mathbf{b}_k^{c,(i)} \qquad (k = 1, \ldots, n^{(i)}) \ . \tag{3.2}$$

This occurs, in particular, in the common case where the coarse variables are collocated with fine variables, $\hat{\mathbf{R}}$ is an injection, and $\mathbf{P}$ is an interpolation (compare with conventions 1, 3, and 4. from Section 2.4). As a consequence, the favorable feature of the CCA coarse grid correction—presented in Proposition 2.5—can be extended in the case of prolongated basis functions: *An error that is in the range of all basis-function matrices vanishes not only at C-points but also at F-points.*

PROPOSITION 3.1. *Let* $\mathbf{B}^{(i)} = \mathbf{P}\mathbf{B}^{c,(i)}$, $i = 1, \ldots, n^c$, *denote the matrices whose columns are the basis functions employed in the* CCA *algorithm where*

$$\mathbf{B}^{c,(i)} = \begin{pmatrix} \mathbf{b}_1^{c,(i)} & \mathbf{b}_2^{c,(i)} & \ldots & \mathbf{b}_{n^{(i)}}^{c,(i)} \end{pmatrix}$$

is the matrix whose columns are the coarse basis functions. Furthermore, let $\mathbf{e}$, *the fine-grid error before the* CCA *coarse-grid correction, satisfy* $\mathbf{e} \in \cap_{i=1}^{n^c} \mathcal{R}\left(\mathbf{B}^{(i)}\right)$, *and assume that* $\mathbf{A}^c$ *is nonsingular. Finally, let* $\hat{\mathbf{R}}\mathbf{P} = \mathbf{I}^c$. *Then* $\mathbf{e}_{new}$, *the error after the coarse-grid correction, vanishes.*

Proof. Since $\mathbf{e}$ can be expressed as a linear combination of the columns of $\mathbf{B}^{(i)}$ for each coarse grid point i, then it can be written as $\mathbf{e} = \mathbf{P}\mathbf{e}^c$, where $\mathbf{e}^c \in \cap_{i=1}^{n^c} \mathcal{R}\left(\mathbf{B}^{c,(i)}\right)$. Thus, (3.2) implies that $\mathbf{A}_i^c \mathbf{e}^c = \mathbf{R}_i \mathbf{A} \mathbf{P} \mathbf{e}^c$ is satisfied for all $i = 1, \ldots, n^c$, and therefore, $\left[\mathbf{A}^c - \mathbf{R}\mathbf{A}\mathbf{P}\right] \mathbf{e}^c = 0$.

Consider now the well-known general error propagation matrix of the coarse-grid correction (with $\mathbf{I}$ denoting the fine identity matrix). Then we have:

$$\begin{aligned} \mathbf{e}_{new} &= \left[\mathbf{I} - \mathbf{P}\left(\mathbf{A}^c\right)^{-1} \mathbf{R}\mathbf{A}\right] \mathbf{e} \\ &= \left[\mathbf{I} - \mathbf{P}\left(\mathbf{A}^c\right)^{-1} \mathbf{R}\mathbf{A}\right] \mathbf{P}\mathbf{e}^c \\ &= \left[\mathbf{P} - \mathbf{P}\left(\mathbf{A}^c\right)^{-1} \mathbf{R}\mathbf{A}\mathbf{P}\right] \mathbf{e}^c \\ &= \mathbf{P}\left(\mathbf{A}^c\right)^{-1} \left[\mathbf{A}^c - \mathbf{R}\mathbf{A}\mathbf{P}\right] \mathbf{e}^c = 0 \, . \end{aligned} \tag{3.3}$$

□

We see that CCA using prolongated basis functions can be viewed as a certain approximation to GCA where we keep the control over the coarse grid sparsity pattern (and thus the control over the computational work.) As a trade-off, if we reduce the stencil (compared to that of GCA), then only errors that lie in a certain subspace of the range of the prolongation are annihilated exactly by the coarse-grid correction, in contrast to GCA, for which all such errors vanish after the coarse-grid correction (cf. equation 3.3).

Inserting the prolongated CBFs into the explicit formula (2.17) for $\mathbf{A}_{ij}^c$ gives

$$\mathbf{A}_{ij}^c = \mathbf{R}_i \mathbf{A} \mathbf{P} \bar{\mathbf{b}}_{k(j)}^{c,(i)} \qquad (k = 1, \ldots, n^{(i)}) \ .$$

If we now choose "delta functions" for the coarse CBFs, we evidently obtain

$$\mathbf{A}^c_{ij} = \mathbf{R}_i \mathbf{A} \mathbf{P}^j \qquad (k = 1, \ldots, n^{(i)}) \ ,$$

where $\mathbf{P}^j$ is the jth column of $\mathbf{P}$. In this case, if we choose an appropriate stencil, we simply recover Petrov-Galerkin coarse approximation, cf. (1.2). In this sense, as noted above, CCA is a generalization of GCA, where we have gained control over the sparsity pattern of the coarse operator at the expense of redundancy in the basis functions, as each coarse point j may have different CBFs associated with it for different rows in $\mathbf{A}^c$. This (possibly) different choice of basis functions at different "collocation" points corresponds to the "generalization" of the collocation idea mentioned above at the end of Section 2.2.

We next note that fine CBFs are themselves prolongations of certain coarse CBFs: Provided that

$$\mathbf{B}^{(i)} = \mathbf{P}\mathbf{B}^{c,(i)}$$

one immediately obtains

$$\bar{\mathbf{B}}^{(i)} = \mathbf{B}^{(i)} \left(\mathbf{M}^{(i)}\right)^{-1} = \mathbf{P}\mathbf{B}^{c,(i)} \left(\mathbf{M}^{(i)}\right)^{-1} = \mathbf{P}\bar{\mathbf{B}}^{c,(i)} \ .$$

We would like to point out that this feature is particulary helpful for an efficient implementation of CCA, since the transformation of the fine basis functions into canonical form "can be done on the coarse grid". Moreover, if the same set of $n^{(i)}$ coarse basis functions is used at every C-point one only needs to solve a small linear $(n^{(i)} \times n^{(i)})$-system *once* in order to accomplish the whole transformation process for all C-points.

EXAMPLE 1 CONTINUED. *Note that on the neighborhood of coarse grid point i, $\Omega^{(i)} = \{i - n^c_x + 1, i - 1, i, i + 1, i + n^c_x - 1\}$, only three coarse basis functions $\mathbf{b}^{c,(i)}_1(x), \mathbf{b}^{c,(i)}_2(x), \mathbf{b}^{c,(i)}_3(x)$ that only depend on x can be linearly independent. This is due to the fact that $\Omega^{(i)}$ only contains three grid points with different x-coordinates, i.e., $\{i - 1, i, i + 1\}$. As a consequence, an arbitrary function $\mathbf{g}(x)$ can be expressed as a linear combination of such three basis functions, e.g. by $1, x, x^2$. Obviously, an analogous statement holds w.r.t. to the y-direction. Hence, if we choose any five coarse basis functions that are linearly independent on $\Omega^{(i)}$, and are of the "additive form" $\mathbf{b}^{c,(i)}_k = \mathbf{g}(x) + \mathbf{h}(y)$, we obtain the following coarse CBFs (supplemented by all $90°$ rotations):*

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}_{2h} \ , \qquad \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}_{2h} \ . \tag{3.4}$$

Prolongating these coarse CBFs to the fine grid, yields the fine CBFs. By Proposition 3.1, any fine-grid error that is given by the prolongation of such additive-form coarse-grid function is eliminated exactly by the CCA coarse-grid correction. This means that choosing $1, x, x^2, y, y^2$ as coarse basis functions does not only make sense for geometrically smooth error components but also for algebraically smooth components as long as a proper (usually matrix-dependent) prolongation is selected.

The related prolongated CBFs using bilinear interpolation (again supplemented by

all 90° *rotations) are:*

$$\begin{bmatrix} 1 & 0.5 & \underline{\mathbf{0}} & 0 & 0 \\ 1 & 0.5 & 0 & 0 & 0 \\ \underline{\mathbf{1}} & 0.5 & \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} \\ 1 & 0.5 & 0 & 0 & 0 \\ 1 & 0.5 & \underline{\mathbf{0}} & 0 & 0 \end{bmatrix}_h , \qquad \begin{bmatrix} -1 & -0.5 & \underline{\mathbf{0}} & -0.5 & -1 \\ -0.5 & 0 & 0.5 & 0 & -0.5 \\ \underline{\mathbf{0}} & 0.5 & \underline{\mathbf{1}} & 0.5 & \underline{\mathbf{0}} \\ -0.5 & 0 & 0.5 & 0 & -0.5 \\ -1 & -0.5 & \underline{\mathbf{0}} & -0.5 & -1 \end{bmatrix}_h .$$

Note that the property exhibited in Example 1, of exact elimination of all errors in the range of prolongation whose mixed derivatives vanish identically on the coarse grid, extends to higher dimensions. This demonstrates the tradeoff afforded by CCA. For a d-dimensional problem, the size of the CCA coarse-grid stencil is $2d+1$, growing linearly with d, as opposed to the exponentially growing 3^d of GCA (assuming the common choice of transfer operators that are represented by 3^d-point stencils as well). The cost one pays is that only a subspace of the range of $\mathbf{P}$ is handled exactly by the coarse grid. With an appropriate choice of this subspace (as done in this example), we obtain a multigrid solver that is nearly as robust as the GCA solver, while retaining small coarse stencils.

We next point out that reducing the stencil size compared to the GCA coarse operator leads to an unfavorable property, namely, that symmetry is not preserved in general.

PROPOSITION 3.2 (Symmetry). *In general, CCA does not preserve symmetry of the operator. From (2.17), we obtain the following condition for symmetry of* $\mathbf{A}^c$*. The coarse operator,* $\mathbf{A}^c$*, produced by CCA is symmetric if, for every pair of coarse grid indices* $\{i,j\}$*,* $j \in \Omega^{(i)}$ *whenever* $i \in \Omega^{(j)}$*, and furthermore,*

$$\mathbf{R}_i \mathbf{A} \bar{\mathbf{b}}^{(i)}_{k(j)} = \mathbf{R}_j \mathbf{A} \bar{\mathbf{b}}^{(j)}_{k(i)} ,$$

where $\bar{\mathbf{b}}^{(i)}_{k(j)}$ *and* $\bar{\mathbf{b}}^{(j)}_{k(i)}$ *are the CBFs employed in computing* $\mathbf{A}^c_{ij}$ *and* $\mathbf{A}^c_{ji}$*, respectively.* A particular case occurs, of course, when $\mathbf{A}$ is symmetric, $\mathbf{R} = \mathbf{P}^T$, and the columns of $\mathbf{P}$ are used as the CBFs, in which case CCA (with the proper stencil) coincides with Galerkin coarsening. It is interesting, in this context, to consider the basis functions associated with bi-linear interpolation:

EXAMPLE 3. *Fix the nine-point stencil (2.20) at coarse grid point* i*, and let the nine canonical basis functions be given by*

$$\bar{\mathbf{b}}^{(i)}_1 = \frac{1}{4}\begin{bmatrix} \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} \\ 0 & 1 & 2 & 1 & 0 \\ \underline{\mathbf{0}} & 2 & \underline{\mathbf{4}} & 2 & \underline{\mathbf{0}} \\ 0 & 1 & 2 & 1 & 0 \\ \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} \end{bmatrix}_h , \quad \bar{\mathbf{b}}^{(i)}_2 = \frac{1}{4}\begin{bmatrix} \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} \\ 2 & 1 & 0 & 0 & 0 \\ \underline{\mathbf{4}} & 2 & \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} \\ 2 & 1 & 0 & 0 & 0 \\ \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} \end{bmatrix}_h ,$$

$$\bar{\mathbf{b}}^{(i)}_6 = \frac{1}{4}\begin{bmatrix} \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} \\ 0 & 0 & 0 & 0 & 0 \\ \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} \\ 2 & 1 & 0 & 0 & 0 \\ \underline{\mathbf{4}} & 2 & \underline{\mathbf{0}} & 0 & \underline{\mathbf{0}} \end{bmatrix}_h , \tag{3.5}$$

with $\bar{\mathbf{b}}^{(i)}_3$*,* $\bar{\mathbf{b}}^{(i)}_4$*, and* $\bar{\mathbf{b}}^{(i)}_5$*, given by successive* 90° *rotations of* $\bar{\mathbf{b}}^{(i)}_2$*, and with* $\bar{\mathbf{b}}^{(i)}_7$*,* $\bar{\mathbf{b}}^{(i)}_8$*, and* $\bar{\mathbf{b}}^{(i)}_9$*, given by successive* 90° *rotations of* $\bar{\mathbf{b}}^{(i)}_6$*. Note that this choice refers to*

bi-linear interpolation of delta functions. Then, with $\mathbf{R}$ *chosen to be the usual full-weighted residual transfer—the transpose of bi-linear interpolation—the CCA coarse operator coincides with the Galerkin coarse operator. Transforming back to low-order polynomial basis functions, (3.5) turn out to be equivalent to* $\{1,(x-x_i),7(x-x_i)^2-(x-x_i)^4\}\times\{1,(y-y_i),7(y-y_i)^2-(y-y_i)^4\}$. *Here,* (x_i,y_i) *is the location of the* i*th coarse variable. Note that here we cannot shift the origin without loss of generality, and therefore it is not true that an error that is globally in the subspace spanned by the test functions* $\{1,x,7x^2-x^4\}\times\{1,y,7y^2-y^4\}$ *is corrected exactly at* C *points (though it remains true for linear functions.)*

4. Numerical Tests. In this section, the efficiency and robustness of the CCA approach is investigated and compared to DCA and GCA. We consider the two-dimensional diffusion equation on the unit square with Dirichlet boundary conditions

$$\begin{aligned} -\left(\,a(x,y)\,u_x(x,y)\,\right)_x-\left(\,b(x,y)\,u_y(x,y)\,\right)_y &= f(x,y) \qquad \left((x,y)\in\Omega=(0,1)^2\right)\ , \\ u(x,y) &= g(x,y) \qquad \left((x,y)\in\partial\Omega\right)\ . \end{aligned}$$

The diffusion problem is discretized by a finite volume method based on arithmetic averages leading to a five-point stencil on a discrete domain Ω_h with regular mesh size $h_x=h_y=h=1/64$. More precisely, the stencil entries for grid point $(x_\mathbf{i},y_\mathbf{i})\in\Omega_h$ read

$$\begin{aligned} W_\mathbf{i} &= -\frac{1}{2h^2}\left(a(x_\mathbf{i}-h,y_\mathbf{i})+a(x_\mathbf{i},y_\mathbf{i})\right)\ , \qquad E_\mathbf{i} = -\frac{1}{2h^2}\left(a(x_\mathbf{i},y_\mathbf{i})+a(x_\mathbf{i}+h,y_\mathbf{i})\right)\ , \\ S_\mathbf{i} &= -\frac{1}{2h^2}\left(b(x_\mathbf{i},y_\mathbf{i}-h)+b(x_\mathbf{i},y_\mathbf{i})\right)\ , \qquad N_\mathbf{i} = -\frac{1}{2h^2}\left(b(x_\mathbf{i},y_\mathbf{i}+h)+b(x_\mathbf{i},y_\mathbf{i}+h)\right)\ , \\ C_\mathbf{i} &= -\left(W_\mathbf{i}+E_\mathbf{i}+S_\mathbf{i}+N_\mathbf{i}\right)\ . \end{aligned}$$

To demonstrate the potential of CCA we consider a collection of (classical) test cases taken from the literature, see [1, 9]. Since an emphasis is laid on the robustness of the different coarse approximations, the suite of test cases covers a variety of possible inhomogeneities including simple vertical jumps, jumps involving corners, jumps that are aligned with the gridlines, jumps that are not aligned with the gridlines, a thin inhomogeneous layer, and complex inhomogeneous domains. In particular we have:

TC1: Vertical jump,	**TC2**: Inhomogeneous strip,
TC3: Inhomogeneous square,	**TC4**: Inhomogeneous diamond,
TC5: Inhomogeneous staircase,	**TC6**: Inhomogeneous L-shaped strip.

The different choices for the diffusion coefficients a,b for **TC1** - **TC6** are specified in Figures 4.1, 4.2, and 4.3.

We use a random initial guess. Right hand side f and boundary condition g are set to zero in all test cases. Hence, the exact solution is known to be zero as well. Thus, the numerical experiments are not limited by machine accuracy and we are able to determine asymptotic convergence factors.

Except for the coarse grid approximation the multigrid components are fixed as follows: We use $V(1,1)$-cycles (i.e. one pre- and one-postrelaxation), six levels, red-black Jacobi relaxation and black-box transfer operators introduced in [10, 11]. Next to DCA we consider four different variants of CCA for the coarse grid approximation:

- **CCA5P: Fine CBFs:** A compact 5-point coarse grid stencil with the five fine CBFs from (2.19),
- **CCA5P: Prol. CBFs:** A compact 5-point coarse grid stencil with the five prolongated coarse CBFs from (3.4),

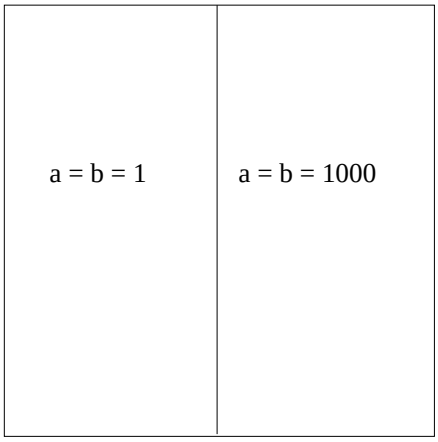

Fig. 4.1 (a) **TC1**.

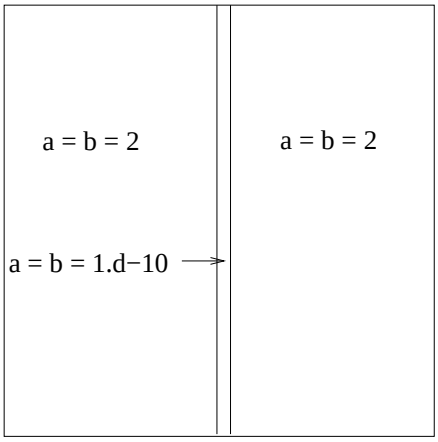

Fig. 4.1 (b) **TC2**.

FIG. 4.1. *Distribution of diffusion coefficients* a, b *for* **TC1**, **TC2**.

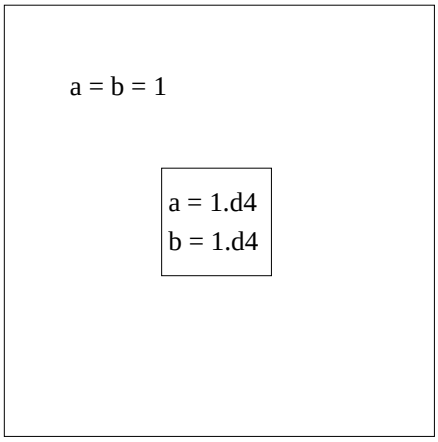

Fig. 4.2 (a) **TC3**.

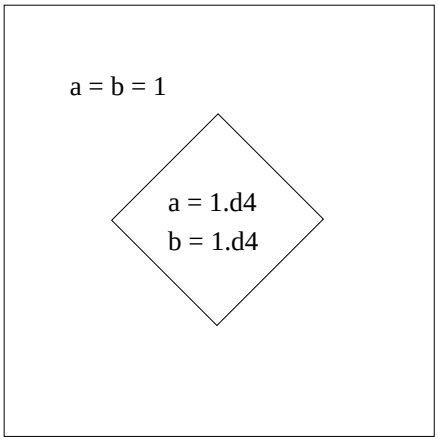

Fig. 4.2 (b) **TC4**.

FIG. 4.2. *Distribution of diffusion coefficients* a, b *for* **TC3**, **TC4**.

- **CCA9P: Fine CBFs:** A compact 9-point coarse grid stencil with the nine fine CBFs from (2.21),
- **CCA9P: Prol. CBFs:** A compact 9-point coarse grid stencil with the nine prolongated coarse CBFs referring to $\{1, x, x^2\} \times \{1, y, y^2\}$. Note that sampling the CBFs from (2.21) on the coarse grid leads to nine delta functions. Hence **CCA9P: Prol. CBFs** corresponds to *prolongated delta functions*.

Please note that **CCA9P: Prol. CBFs** and GCA coincide, so it is equivalent to the classical black box multigrid. In case of **CCA5P** we may lose symmetry for the coarse grid approximations due to Proposition 3.2. Then, the matrix-dependent restriction $\mathbf{R}$ is based on $(\mathbf{A}^c)^T$, compare with [11]. Summarizing, we consider three 5-point coarse approximations and two 9-point coarse approximations.

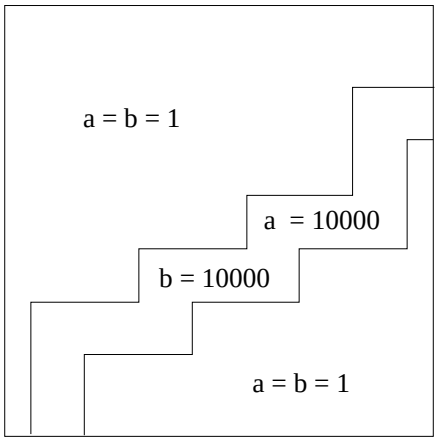

Fig. 4.3 (a) **TC5**.

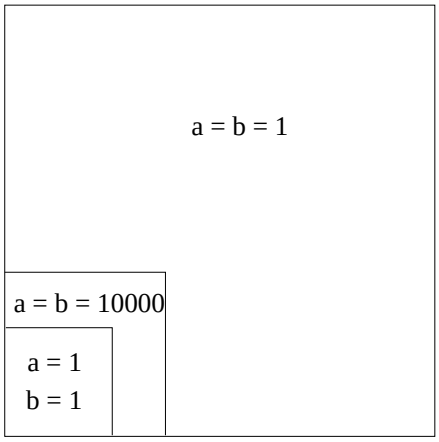

Fig. 4.3 (b) **TC6**.

FIG. 4.3. *Distribution of diffusion coefficients* a, b *for* **TC5**, **TC6**.

TABLE 4.1
Asymptotic residual reduction factors.

Coarse Approximation	TC1	TC2	TC3	TC4	TC5	TC6
DCA	0.264	0.685	DIV.	0.999	0.616	DIV.
CCA5P: Fine CBFs	0.100	0.139	0.981	0.999	0.185	0.282
CCA5P: Prol. CBFs	0.100	0.133	0.249	0.999	0.166	0.293
CCA9P: Fine CBFs	0.067	0.084	0.981	0.997	0.163	0.232
CCA9P: Prol. CBFs / GCA	0.068	0.080	0.218	0.122	0.083	0.212

The asymptotic residual reduction factors shown in Table 4.1 are given by

$$\sqrt[5]{\frac{\|\mathbf{r}(\mathbf{u}^{20})\|_\infty}{\|\mathbf{r}(\mathbf{u}^{15})\|_\infty}} \ ,$$

where $\mathbf{r}(\mathbf{u}^m)$ refers to the residual vector associated with the approximation $\mathbf{u}^m$ of the exact solution of the underlying linear system after m multigrid $V(1,1)$-cycles. The convergence histories for **TC1** - **TC6** are shown in Figures 4.4 - 4.9.

As expected, DCA performs worst since it is not designed to cope with difficult diffusion problems including huge jumps. On the other hand CCA9P: Prol. CBFs / GCA turns out to be the best coarse approximation, which is no surprise either, since its adequacy is well-known from the literature for such kind of problems. Prolongated CBFs are superior to the (geometrically smooth) fine CBFs, which justifies the motivation given in Section 3: The use of just *geometrically* smooth fine CBFs is not sufficient in general situations as observed for **TC3** and **TC4**. It is interesting to note that in many cases CCA5P: Prol. CBFs performs nearly as well as CCA9P: Prol. CBFs / GCA (there was only one exception, cf. (**TC4**). This demonstrates that is indeed possible to reduce the stencil size of GCA while maintaining a robust multigrid algorithm.

5. Conclusions and outlook. A new approach—CCA—for the construction of coarse approximations in multigrid has been presented. It turns out that the well-known Galerkin coarse approximation can be considered as a special case of CCA. The advantage of CCA over GCA is that it allows explicit control over the sparsity

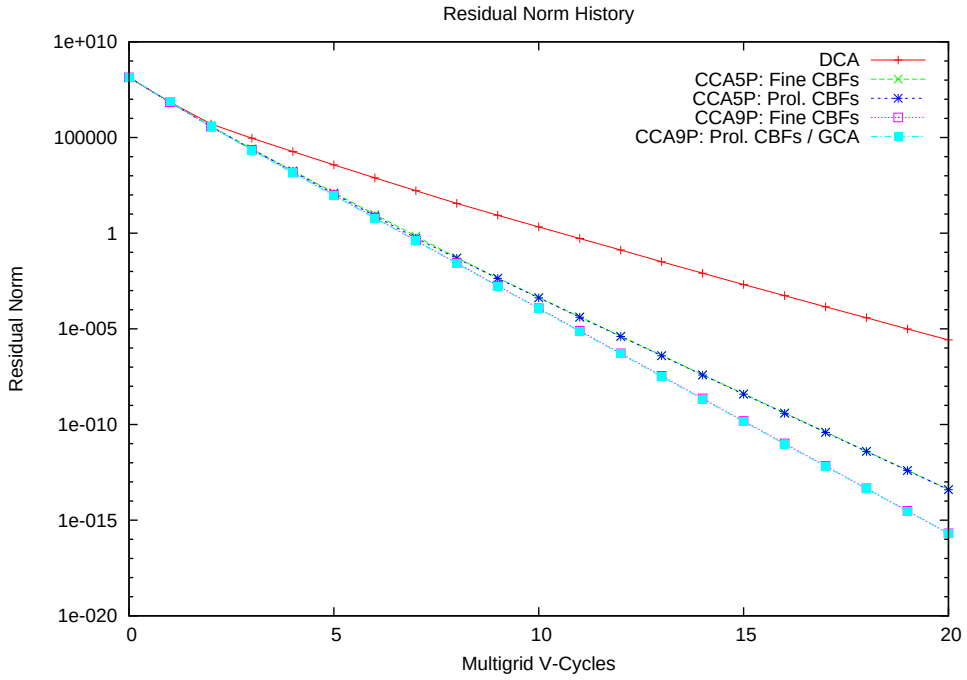

FIG. 4.4. *Convergence history for* **TC1***: Vertical jump.*

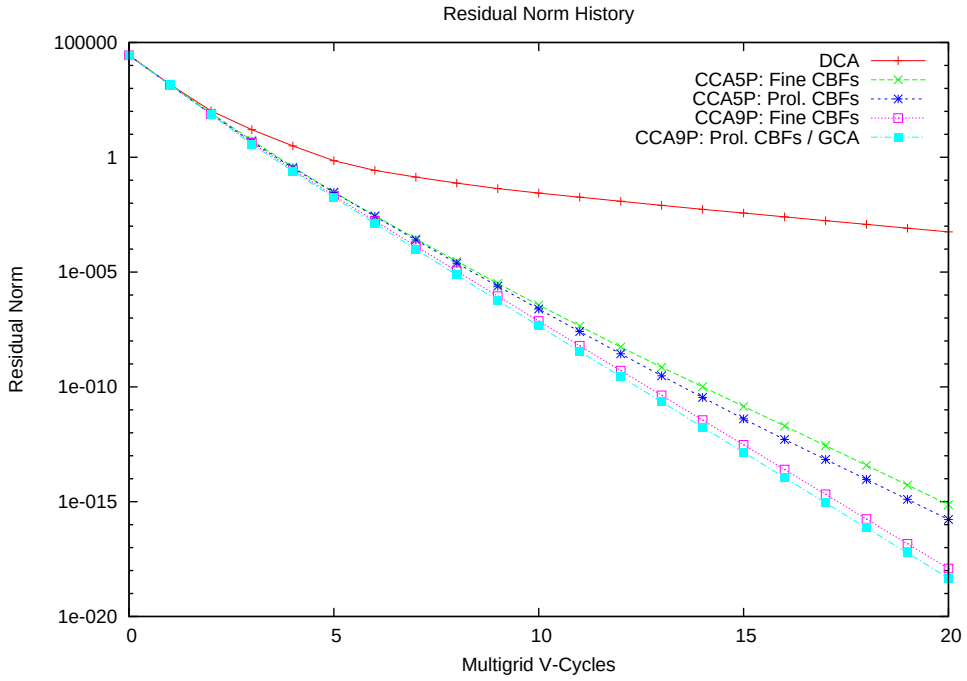

FIG. 4.5. *Convergence history for* **TC2***: Inhomogeneous strip.*

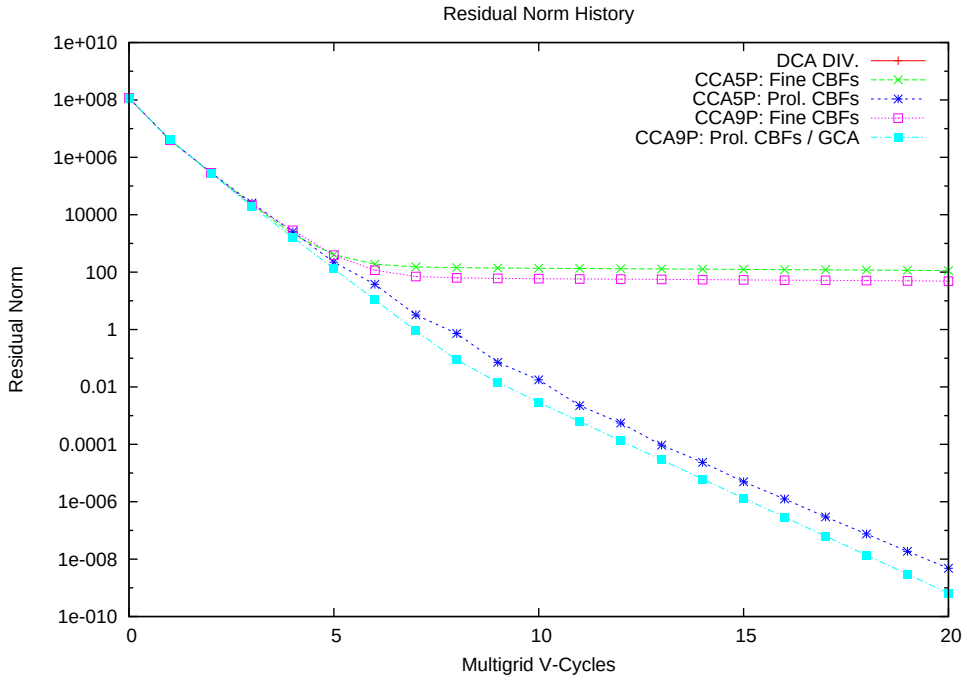

FIG. 4.6. *Convergence history for* **TC3***: Inhomogeneous square. The multigrid algorithm based on DCA diverged.*

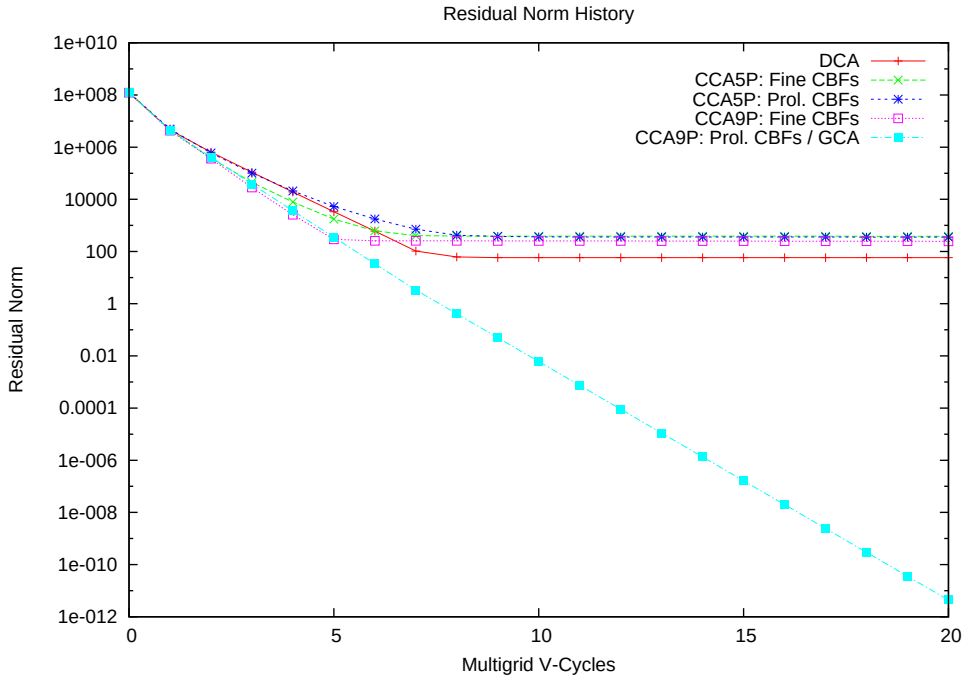

FIG. 4.7. *Convergence history for* **TC4***: Inhomogeneous diamond.*

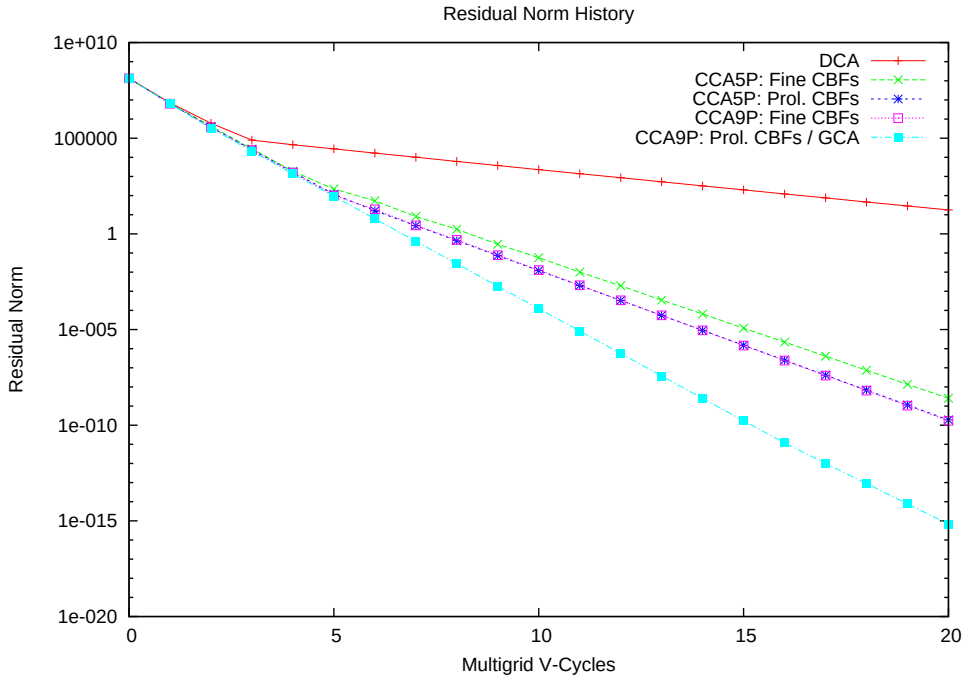

FIG. 4.8. *Convergence history for* **TC5***: Inhomogeneous staircase.*

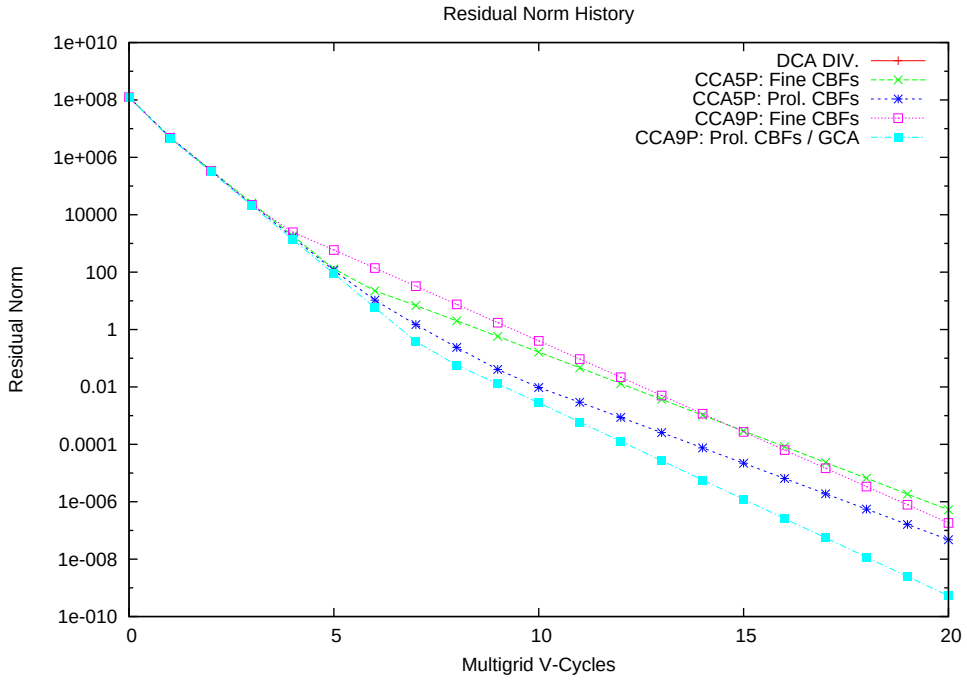

FIG. 4.9. *Convergence history for* **TC6***: Inhomogeneous L-shaped strip. The multigrid algorithm based on DCA diverged.*

pattern of the coarse approximation matrix, independently of the transfer operators which is particularly worthwhile for parallel computing. It has been demonstrated by a selection of test cases that the robustness of GCA can often be maintained by CCA without increasing the sparsity pattern on coarser levels. This desirable feature should be particularly helpful for three-dimensional problems and systems of partial differential equations. The application of CCA to such problems is a natural and very promising next step which should be investigated. Another obvious direction for future research is the evaluation of nonsymmetric problems, since CCA does not preserve but also *does not require* symmetry.

CCA has been introduced in a purely algebraic way (although geometric information can and should be used whenever possible). Hence, the evaluation of CCA for unstructured grids is a very interesting issue which warrants further research and a proper adjustment of the algorithm—in particular, an appropriate choice for the basis functions in this case has to be identified.

Finally, we would like to point out that the governing equations (2.14) can be used to construct the coarse approximation (as it is done in CCA so far) *and* the restriction $\mathbf{R}$ at the same time. Then the nonzero entries of the ith row of $\mathbf{R}$ have to be determined by equations (2.14) as well increasing the number of basis functions for which the fine-to-coarse defect correction vanishes, compare with (2.8). Preliminary numerical tests indicate a very promising behavior of this generalization.

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